

CALCULUS II TUTORIAL (3) : SOLUTIONS

①

$$(a) \left(\frac{3n+1}{3n-1} \right)^n = \left(\frac{1 + \frac{1}{3n}}{1 - \frac{1}{3n}} \right)^n \approx \left(1 + \frac{2}{3n} \right)^n \rightarrow e^{2/3} \text{ as } n \rightarrow \infty$$

using $\left(1 + \frac{x}{n} \right)^n \rightarrow e^x$ and $\left(1 - \frac{1}{3n} \right)^{-1} \approx \left(1 + \frac{1}{3n} \right)^{-1}$ for large n .

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln \left(1 + \frac{x}{n} \right)^n &= \lim_{n \rightarrow \infty} \left[\frac{\ln \left(1 + \frac{x}{n} \right)}{1/n} \right] = \lim_{n \rightarrow \infty} \left[\frac{-\frac{x}{n^2} \frac{1}{1+x/n}}{-1/n^2} \right] \\ &= \lim_{n \rightarrow \infty} \left(\frac{x}{1+x/n} \right) = x \Rightarrow \left(1 + \frac{x}{n} \right)^n \rightarrow e^x \quad \uparrow \text{L'Hôpital} \end{aligned}$$

This has been proved in lectures.]

$$(b) \lim_{n \rightarrow \infty} \left[\frac{n^2}{2n-1} \sin \frac{1}{n} \right] = \lim_{n \rightarrow \infty} \left[\frac{n}{2} \sin \frac{1}{n} \right] = \lim_{n \rightarrow \infty} \left[\frac{-\frac{1}{n^2} \cos \frac{1}{n}}{-\frac{2}{n^2}} \right] = \frac{1}{2}$$

using L'Hôpital's rule, $\cos \frac{1}{n} \approx 1$, $2n-1 \approx 2n$ for large n

$$(c) \frac{1}{n} \int_1^n \frac{1}{x} dx = \frac{1}{n} [\ln x]_1^n = \frac{\ln n}{n} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ since}$$

$$\lim_{n \rightarrow \infty} \left[\frac{\ln n}{n} \right] = \lim_{n \rightarrow \infty} \left[\frac{1/n}{1} \right] = 0 \text{ (using L'Hôpital)}$$

②

$$(a) \sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right) = \left[5 + \frac{5}{2} + \frac{5}{4} + \dots \right] + \left[1 + \frac{1}{3} + \frac{1}{9} + \dots \right] = \frac{5}{1-1/2} + \frac{1}{1-1/3} = \frac{23}{2}$$

using $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ with $a=5, r=1/2$ and $a=1, r=1/3$.

$$(b) \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{4n-3} - \frac{1}{4n+1} \right] = \left(1 - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \dots = 1$$

$$(c) \sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2} = \sum_{n=1}^{\infty} 5 \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right] = 5$$

since $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right] = \left(1 - \frac{1}{9} \right) + \left(\frac{1}{9} - \frac{1}{25} \right) + \dots = 1$

③

$$(a) \frac{\ln(n+1)}{\sqrt{n}} \approx \frac{\ln n}{\sqrt{n}} \text{ (large } n) = 2 \frac{\ln(\sqrt{n})}{\sqrt{n}} \rightarrow 0 \text{ since } \frac{\ln n}{n} \rightarrow 0$$

[Last result was proved in solution 1(c)]

$$(b) \frac{n!}{n^n} = \frac{n(n-1)(n-2)\dots 2 \cdot 1}{n \cdot n \cdot n \dots n \cdot n} = 1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \frac{2}{n} \cdot \frac{1}{n} < \frac{1}{n}$$

since all terms before last are less than 1.

$$\text{But } \frac{1}{n} \rightarrow 0, \text{ so } \frac{n!}{n^n} \rightarrow 0$$

$$(c) \lim_{n \rightarrow \infty} \left[n \ln \left(1 + \frac{7}{n} \right) \right] = \lim_{n \rightarrow \infty} \frac{1 + \frac{7}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left[\frac{-\frac{7}{n^2}}{-\frac{1}{n^2}} \right] = 7 \Rightarrow \left(1 + \frac{7}{n} \right)^n \rightarrow e^7$$

$$(d) n - \sqrt{n^2 - n} = n - n \left[1 - \frac{1}{n} \right]^{\frac{1}{2}} \approx n - n \left[1 - \frac{1}{2n} \right] \approx \frac{1}{2} \text{ for large } n$$

④ Using binomial thm: $(1-x)^{1/2} \approx 1 - \frac{1}{2}x$ for $x \gg 1$.

$$(a) \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n = 1 + \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}} \right)^2 + \dots = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 2 + \sqrt{2}$$

$$\text{using } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ with } a=1, r=\frac{1}{\sqrt{2}}$$

$$(b) \frac{x^n}{n!} \rightarrow 0 \text{ for all } x \Rightarrow \frac{1000^n}{n!} \rightarrow 0 \Rightarrow \frac{n!}{1000^n} \rightarrow \infty \Rightarrow \sum_{n=0}^{\infty} \frac{n!}{1000^n} \text{ diverge}$$

$$\left[\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \dots M \cdot (M+1)(M+2) \dots n} \text{ where } M > |x| \text{ (Proved in lectures)} \right]$$

$$\leq \frac{|x|^n}{M! M^{n-M}} = \frac{|x|^n M^M}{M! M^n} = \frac{M^M}{M!} \left(\frac{|x|}{M} \right)^n \rightarrow 0 \text{ since } \frac{|x|}{M} < 1$$

$$(c) \sum_{n=1}^{\infty} \ln \left(\frac{n}{n+1} \right) = \ln \frac{1}{2} + \ln \frac{2}{3} + \dots + \ln \frac{n}{n+1} = \ln \left(\frac{1}{n+1} \right)$$

$$\text{As } n \rightarrow \infty, \frac{1}{n+1} \rightarrow 0 \Rightarrow \ln \frac{1}{n+1} \rightarrow -\infty \Rightarrow \sum_{n=1}^{\infty} \ln \left(\frac{n}{n+1} \right) \text{ diverges}$$

⑤

$$\text{Area} = 2 \times \frac{1}{2} \pi \left(\frac{1}{2} \right)^2 + 4 \times \frac{1}{2} \pi \left(\frac{1}{4} \right)^2 + 8 \times \frac{1}{2} \pi \left(\frac{1}{8} \right)^2 + \dots$$

$$= \frac{\pi}{4} + \frac{\pi}{8} + \frac{\pi}{16} + \dots = \frac{\pi/4}{1-1/2} = \frac{\pi}{2}$$